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Approximation by $\phi(ax)\, L(A, x)$ on Finite Point Sets

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Let $X = \{x_1, ..., x_N\}$ be a finite subset of the real line, $x_1 < \cdots < x_N$. Let ϕ be a continuous function on the real line and $\{\psi_1, ..., \psi_n\}$ a Chebyshev set on X, $n \leq N$. Define $L(A, x) = \sum_{k=1}^{n} a_k \psi_k(x)$, $F(A, x) = \phi(a_0 x) L(A, x)$. Let $\| \ \|$ be a given norm on the functions on X. Let G be a family of functions containing $\{F(A, \cdot)\}$. The approximation problem is: Given a function f on X, find $g^* \in G$ for which $\| f - g \|$ attains its infimum $\rho(f)$ over $g \in G$. Such an element g^* is called a best approximation. In this note we consider the existence of best approximations.

It is well known that a necessary and sufficient condition that every function on X have a best approximation from G is that G is closed. We, therefore, seek to find the family $\bar{F}$, the closure of $\{F(A, \cdot)\}$. This $\bar{F}$ has the property that a best approximation from it always exists and it is the smallest family G containing $\{F(A, \cdot)\}$ with this property. Characterizing $\bar{F}$ involves two steps. We must show that $\bar{F}$ contains all limits of sequences from $\{F(A, \cdot)\}$. We must also show that each element of $\bar{F}$ is a limit of a sequence from $\{F(A, \cdot)\}$. Since we may not know $\bar{F}$ ahead of time, we will first consider limits of bounded sequences from $\{F(A, \cdot)\}$ and later see if every element of a family containing them is a limit of a bounded sequence.

Such an analysis has already been carried out for the case $\phi(x) = \exp(x)$ by the author [1].

It will be useful to have a norm on the coefficient vector of L, or equivalently, a seminorm on the parameter vector. Define

$$\| A \| = \max\{| a_i | : 1 \leq i \leq n\}.$$

As the first part of our analysis, we consider the behavior of bounded sequences from $\{F(A, \cdot)\}$. Without loss of generality we will use the Chebyshev norm and consider

$$\| F(A^k, \cdot) \|_X \leq M. \tag{1}$$

The sequence $\{a_0^k\}$ may not be bounded. However, as $[-\infty, \infty]$ is compact,

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the sequence has a limit point a_0^0 in $[-\infty, \infty]$. By taking a subsequence if necessary, we can assume that $\{a_0^k\} \to a_0^0$ and in the remainder of the paper this will be assumed.

LEMMA 1. *Let $\{F(A^k, \cdot)\}$ be a bounded sequence. Let a_0^0 be finite and nonzero. Let ϕ not vanish except possibly at zero. Then $\{F(A^k, \cdot)\} \to F(A^0, \cdot)$.*

Proof. There are at least n points of X at which $\phi(a_0 x)$ does not vanish, assume without loss of generality they are $x_1, \ldots, x_n$. Let

$$\mu = \min\{|\phi(a_0^0 x_i)| : i = 1, \ldots, n\}.$$

There exists K such that for $k > K$,

$$|\phi(a_0^k x_i)| \geq \mu/2 \qquad i = 1, \ldots, n,$$

hence

$$|L(A^k, x_i)| \leq 2M/\mu \qquad i = 1, \ldots, n, k > K.$$

It follows that $\{\| A^k \|\}$ is bounded and so the limit $(a_1^0, \ldots, a_n^0)$ of $(a_1^k, \ldots, a_n^k)$ is finite. We have $\phi(a_0^k x) \to \phi(a_0^0 x)$, $L(A^k, \cdot) \to L(A^0, \cdot)$, hence $F(A^k, \cdot) \to F(A^0, \cdot)$.

It is easily seen that the case where $a_0^0 = 0$ and $\phi(0) \neq 0$ is also taken care of by Lemma 1.

LEMMA 2. *Let $\{F(A^k, \cdot)\}$ be a bounded sequence, Let $a_0^0 = 0$ and $\phi(x) = \sigma x^m + O(|x|^{m+1})$, $\sigma \neq 0$. Then $\{F(A^k, \cdot)\}$ has an accumulation point of the form $x^m L(A, x)$.*

Proof. Assume without loss of generality that $a_0^k > 0$. There exists K such that

$$|\phi(a_0^k x)| \geq |\sigma(a_0^k x)^m|/2 \qquad k > K, x \in X. \tag{2}$$

Suppose that $\{\|(a_0^k)^m A^k\|\}$ was unbounded, then by taking a subsequence if necessary we can assume it tends to infinity, and by a standard result in linear approximation, a variant of which appears in the text of Rice [2, p. 24].

$$\| L((a_0^k)^m A^k, \cdot)\| = \|(a_0^k)^m L(A^k, \cdot)\| \to \infty$$

and by (2)

$$\| F(A^k, x)\| = \| \phi(a_0^k x) L(A^k, x)\| \geq \| \sigma(a_0^k x)^m L(A^k, x)\| \to \infty.$$

It follows that $(a_0^k)^m A^k$ is bounded and has a limit point A, assume it converges to A. Then

$$\begin{aligned} F(A^k, x) &= [\sigma(a_0^k x)^m + O(| a_0^k x |^{m+1})] L(A^k, x) \\ &= \sigma x^m L((a_0^k)^m A^k, x) + O(| a_0^k |^{m+1}) L(A^k, x) \\ &\to \sigma x^m L(A, x) + 0. \end{aligned}$$

LEMMA 3. *Let* $\phi(0) = \sigma x^m + O(| x |^{m+1})$, $\sigma \neq 0$ *and* A *be finite There exists a sequence* $\{F(A^k, x)\} \to x^m L(A, x)$.

Proof. Let $F(A^k, x) = \phi(x/k) L(k^m A, x)/\sigma$, then

$$\begin{aligned} F(A^k, x) &= (x^m/k^m) L(k^m A, x) + O((x/k)^{m+1}) L(k^m A, x) \\ &= x^m L(A, x) + O(1/k). \end{aligned}$$

The remaining possibility is that $| a_0^0 | = \infty$. What happens in this case varies with each ϕ.

EXPONENTIAL TYPE ϕ

As mentioned previously, the case where $\phi(x) = \exp(x)$ has already been studied [1]. We develop a theory to be applied to the cases $\phi(x) = \cosh(x)$ and $\phi(x) = \sinh(x)$.

LEMMA 4. *Let* $\phi(ax)/\phi(ay) \to 0$ *for* $0 \leq x < y$ *as* $a \to \infty$. *Let* $x_1 \geq 0$, $a_0^0 = \infty$. *Then* $\{F(A^k, \cdot)\} \to 0$ *on all but* $x_{N-n+1}, \ldots, x_N$.

Proof. We can suppose without loss of generality that $\| A^k \| \neq 0$ for all k. Define $B^k = A^k/\| A^k \|$, then $\| B^k \| = 1$. $(b_1^k, \ldots, b_n^k)$ has an accumulation point $(b_1, \ldots, b_n)$, assume convergence occurs. As $\| B \| = 1$, $L(B, \cdot)$ is nonzero on at least one of $x_{N-n+1}, \ldots, x_N$. Assume without loss of generality it is nonzero on x_N. Let $i < N - n + 1$ and consider

$$r_i^k = \frac{F(A^k, x_i)}{F(A^k, x_N)} = \frac{\phi(a_0^k x_i)}{\phi(a_0^k x_N)} \cdot \frac{L(A^k, x_i)}{L(A^k, x_N)} = \frac{\phi(a_0^k x_i)}{\phi(a_0^k, x_N)} \cdot \frac{L(B^k, x_i)}{L(B^k, x_N)}.$$

The ratio of ϕ's tends to zero by hypothesis and the ratio of L's tends to $L(B, x_i)/L(B, x_N)$; hence $r_i^k \to 0$. But $| F(A^k, x_N)| < M$, so $F(A^k, x_i) \to 0$.

LEMMA 5. *Let* ϕ *have no zeros for sufficiently large arguments. Let* $\phi(ax)/\phi(ay) \to 0$ *for* $0 \leq x < y$ *as* $a \to \infty$. *Let* $x_1 > 0$, *and* $x_1 = 0$ *or* $\phi(0) \neq 0$. *Given constants* $y_{N-n+1}, \ldots, y_N$ *there exists a sequence* $\{F(A^k, \cdot)\}$

such that $F(A^k, x_i) = y_i$, $i = N - n + 1,..., N$ *and* $F(A^k, x_i) \to 0$ *for* $i \leqslant N - n$.

Proof. The lemma is obvious in the case all of $y_{N-n+1},..., y_N$ are zero so we assume at least one is nonzero.

There exists K such that for $k \geqslant K$

$$\phi(kx_i) \neq 0 \qquad i = N - n + 1,..., N.$$

Assume without loss of generality that $K = 1$. Let $a_0^k = k$. As $\{\psi_1,..., \psi_n\}$ is a Chebyshev set on X, there exists $\{a_1^k,..., a_n^k\}$ such that

$$L(A^k, x_i) = y_i/\phi(kx_i) \qquad i = N - n + 1,..., N,$$

then

$$F(A^k, x_i) = y_i \qquad i = N - n + 1,..., N.$$

Arguing as in the previous lemma, we get $F(A^k, x_i) \to 0$ for $i \leqslant N - n$.

Let F^+ be the set of functions zero except on $\{x_{N-n+1},..., x_N\}$.

EXAMPLE 1. Let $x_1 \geqslant 0$. The closure of

$$F = \{\cosh(a_0x)\, L(A, x)\} \text{ is } F \cup F^+.$$

Proof. By evenness of cosh we can assume that $a_0 \geqslant 0$. Lemmas 1 and 4 ensure that any bounded sequence from F has an accumulation point in $F \cup F^+$. Lemma 5 ensures that every element of $F \cup F^+$ is the limit of a sequence from F.

EXAMPLE 2. Let $x_1 \geqslant 0$. The closure of $F = \{\sinh(a_0x)\, L(A, x)\}$ is $F \cup F^+ \cup \{xL(A, x)\}$.

Proof. By oddness of sinh we can assume that $a_0 \geqslant 0$. Lemmas 1, 2, 4 ensure that any bounded sequence from F has an accumulation point in $F \cup F^+ \cup \{xL(A, x)\}$. Lemmas 3 and 5 ensure that every element of $F \cup F^+ \cup \{xL(A, x)\}$ is the limit of a sequence from F.

NEGATIVE EXPONENTIAL TYPE ϕ

LEMMA 6. *Let* $\phi(ax)/\phi(ay) \to 0$ *for* $0 \leqslant y < x$ *as* $a \to \infty$. *Let* $x_1 \geqslant 0$, *Let* $a_0^0 = \infty$, *then* $\{F(A^k, \cdot)\} \to 0$ *on all but* $x_1 ..., x_n$.

The proof is similar to the proof of Lemma 4.

LEMMA 7. *Let* ϕ *have no zeros for sufficiently large finite arguments. Let*

$\phi(ax)/\phi(ay) \to 0$ for $0 \leqslant y < x$ as $a \to \infty$. Let $x_1 \geqslant 0$, and $x_1 > 0$ or $\phi(0) \neq 0$. Given constants $y_1, \ldots, y_n$ there exists a sequence $\{F(A^k, \cdot)\}$ such that $F(A^k, x_i) = y_i$, $i = 1, \ldots, n$ and $F(A, x_i) \to 0$ for $i > n$.

The proof is similar to the proof of Lemma 5. Let F^- be the set of functions zero except on $\{x_1, \ldots, x_n\}$.

EXAMPLE 3. Let $x_1 \geqslant 0$. The closure of

$$F = \{\operatorname{sech}(a_0 x) L(A, x)\}$$

is $F \cup F^-$

Proof. As sech is even, we can assume $a_0 \geqslant 0$. Lemmas 1 and 6 ensure that any bounded sequence from F has an accumulation point in $F \cup F^-$. Lemma 7 ensures that every element of $F \cup F^-$ is the limit of a sequence from F.

An identical result holds for $\phi(x) = \exp(-x^2)$.

BOUNDED ϕ

We consider the case where ϕ is continuous at $-\infty$ and $+\infty$.

LEMMA 8. *Let $x_1 \geqslant 0$. Let ϕ be continuous and nonzero at ∞. Let $\{F(A^k, \cdot)\}$ be bounded and $\{A^k\} \to A^0$. Let $a_0^0 = +\infty$ then $\{F(A^k, \cdot)\}$ has as an accumulation point a function of the type $\{L(A, \cdot)\}$ on $\{x_1, \ldots, x_N\} \sim \{0\}$.*

Proof. There exists K such that for $k \geqslant K$,

$$|\phi(a_0^k x_i)| > |\phi(\infty)|/2 \qquad x_i > 0.$$

By (1)

$$|\phi(a_0^k x_i)\, L(A^k, x_i)| < M,$$

hence

$$|L(A^k, x_i)| < 2M/|\phi(\infty)| \qquad i > 1, k \geqslant K.$$

It follows that $\|A^k\|$ is bounded and $(a_1^k, \ldots, a_n^k)$ has a finite limit point $(a_1^0, \ldots, a_n^0)$. Assume convergence occurs. Then

$$F(A^k, x_i) \to \phi(\infty)\, L(A^0, x_i) \qquad x_i > 0.$$

EXAMPLE 4. Let $x_1 > 0$. The closure of $F = \{\arctan(a_0 x)\, L(A, x)\}$ is $F \cup \{xL(A, x)\} \cup \{L(A, \cdot)\}$.

Proof. By oddness of arctan we can assume $a_0 \geqslant 0$. By Lemmas 1, 2, and 8, a bounded seqeunce from F has an accumulation point in the given set.

By Lemma 3 and simple arguments similar to those of Lemma 8 any function of the form $xL(A, x)$ or $L(A, \cdot)$ is a limit of elements of F.

Exactly the same result holds for $\phi(x) = \tanh(x)$.

ARGUMENTS RESTRICTED TO A CLOSED SET

Some functions ϕ which we might wish to consider are defined and continuous only on a closed finite interval, causing us to restrict the parameter a_0 to a closed finite interval. For example the functions arcsin, arcos, and arctanh are only defined on $[-1, 1]$. The case where a_0 is restricted to a closed finite interval I containing 0 is handled by Lemmas 1, 2, 3. We get the closure of $F = \{\phi(a_0x)\, L(A, x) : a_0 \in I\}$ being F if $\phi(0) \neq 0$ and $F \cup \{x^m L(A, x)\}$ if $\phi(x) = \sigma x^m + O(|x|^{m+1})$.

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